3-1-1973

Some Observations on 'Optimal' Economic Growth and Exhaustible Resources

Tjalling C. Koopmans

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 356

SOME OBSERVATIONS ON 'OPTIMAL' ECONOMIC GROWTH AND EXHAUSTIBLE RESOURCES

Tjalling C. Koopmans

March 28, 1973

SOME OBSERVATIONS ON 'OPTIMAL' ECONOMIC GROWTH AND EXHAUSTIBLE RESOURCES*

by

Tjalling C. Koopmans**

1. Introduction

It is a principal theme of Irving Fisher's [1930] classical work "The Theory of Interest" that, in a competitive market over time, the real rate of interest is determined by the interacting forces of consumer's intertemporal preferences and the opportunities to shift goods across time offered by technology and resource supply. It will serve as an introduction to the present paper to recall his illustrations (in Chapter VIII) of the effects of technology and of resource availability by a few striking examples, of which I cite two. In the "sheep example," the flock of sheep has a natural annual rate of increase of 10%. Assuming other inputs to be abundant and costless, the rate of interest (with sheep as numéraire at all times) then also equals 10% p.a. In contrast, in the "hardtack example," a group of sailors is stranded on a barren island, each with a supply of nonperishable hardtack. In that numéraire, the rate of interest is zero. In these simple settings, the statements describe

---

*The research described in this paper is supported under grants from the National Science Foundation and from the Ford Foundation.

**I am indebted to Robert Dorfman, William Nordhaus and Herbert Scarf for valuable comments.

the outcome of competitive trades between owners of sheep or of hardtack, regardless of the intertemporal preferences of the trading parties.

In the models of optimal growth theory a social intertemporal preference structure is specified exogenously in the form of a social welfare functional. In many models this is taken to be of the form

$$U = \int_0^T e^{-\rho t} u(c_t) dt \ , \tag{1.1}$$

where c_t denotes consumption at time t , $u(c)$ a strictly increasing and strictly concave function giving the utility flow arising from a consumption flow c , ρ a nonnegative continuous-time discount rate applied to utility, and $e^{-\rho t}$ the corresponding discount factor for time t . Then we can again take the consumption good as numéraire. Denoting the path optimal under given constraints by $\hat{c}_t$, the corresponding discount factor for consumption is

$$e^{-\rho t} \frac{u'(\hat{c}_t)}{u'(\hat{c}_0)} \ , \tag{1.2}$$

which in turn implies a continuous-time interest rate i_t as a function of time, defined likewise with the consumption good as the numéraire. It follows that any constraint on that interest rate i_t arising from technology or resource availability must give rise also to a condition on the optimal path $\hat{c}_t$.

In this note I wish to contrast the optimal paths obtained for the hardtack example, to be renamed the exhaustible resource model, and for a slight generalization of the sheep example, renamed the capital model.

We shall make this comparison both for $\rho = 0$, and for positive ρ . We occasionally place the word "optimal" in quotes as a reminder that the optimality concept is relative to a particular choice of ρ , of $u(\cdot)$, and indeed of all other specifications of the models examined.

In these inferences the rate of interest i_t will not occur explicitly. It was brought up in the foregoing remarks only to point out that the contrast we are examining—like so much else in capital theory—was already noted and analyzed in a market context in Fisher's work.

2. The Capital Model

For this, Ramsey's [1928] model with discounting, we can be very brief because it has been thoroughly examined and discussed in the literature. An additional variable, the capital stock k_t , is introduced, and output $g(k_t)$ from that capital stock is at all times to be optimally divided between consumption c_t and net investment $\dot{k}_t$. The horizon is extended to $T = \infty$. The problem then becomes that of maximizing the welfare functional

$$(2.1) \qquad U = \int_0^\infty e^{-\rho t} u(c_t) dt$$

subject to

$$(2.2) \quad c_t, k_t \geqq 0 , \quad c_t + \dot{k}_t = g(k_t) \quad \text{for} \quad t \geqq 0 , \quad 0 < k_0 \quad \text{(given)}.$$

The utility and production functions are independent of time and satisfy

$$(2.3) \qquad u'(c) > 0 \ , \quad u''(c) < 0 \quad \text{for} \quad c > 0 \ ,$$

$$(2.4) \quad g(0) = 0 \ , \quad g'(k) > 0 \ , \quad g'(\hat{k}) = 0 \ , \quad g''(k) < 0 \quad \text{for all} \quad k \geqq 0 \quad \text{and some} \quad \hat{k} > 0 \ .$$

While neither resources nor labor appear explicitly, their presence in the background is implicit in the strict concavity of $g(\cdot)$ and in the assumption of a finite saturation level $\hat{k}$ for capital. The sheep fit into this model both as capital and as a source of food and clothing, provided labor and land are limited and, indeed, constant over time.

The well-known characteristics of the unique optimal path in this model are exhibited in Figure 1, which is more fully explained in Koopmans [1967]. In the case of a low initial capital stock k_0, the optimal capital path $\hat{k}_t$ climbs monotonically (starting up more steeply the smaller ρ is) and approaches an asymptotic level $\hat{k}(\rho)$, which is higher as ρ is smaller. Initially optimal consumption $\hat{c}_t$ is higher as ρ is larger, but the asymptotic level $\hat{c}(\rho)$ (which can be read from the diagram at left) is again higher as ρ is smaller. For $\rho = 0$ the asymptotic capital stock $\hat{k}(0) \equiv \hat{k}$ achieves the highest sustainable consumption flow $\hat{c}(0) \equiv \hat{c}$. <u>Hence, in the capital model, discounting of future utilities as compared with no discounting favors consumers in a nearby future. It also levels off consumption over the rest of the future sooner, and at a level lower than what would ultimately be attainable by greater and longer initial sacrifice of consumption. These effects are stronger, the higher is</u> ρ.

The construct of capital saturation, so outside the range of experience, helps overcome a mathematical difficulty connected with the

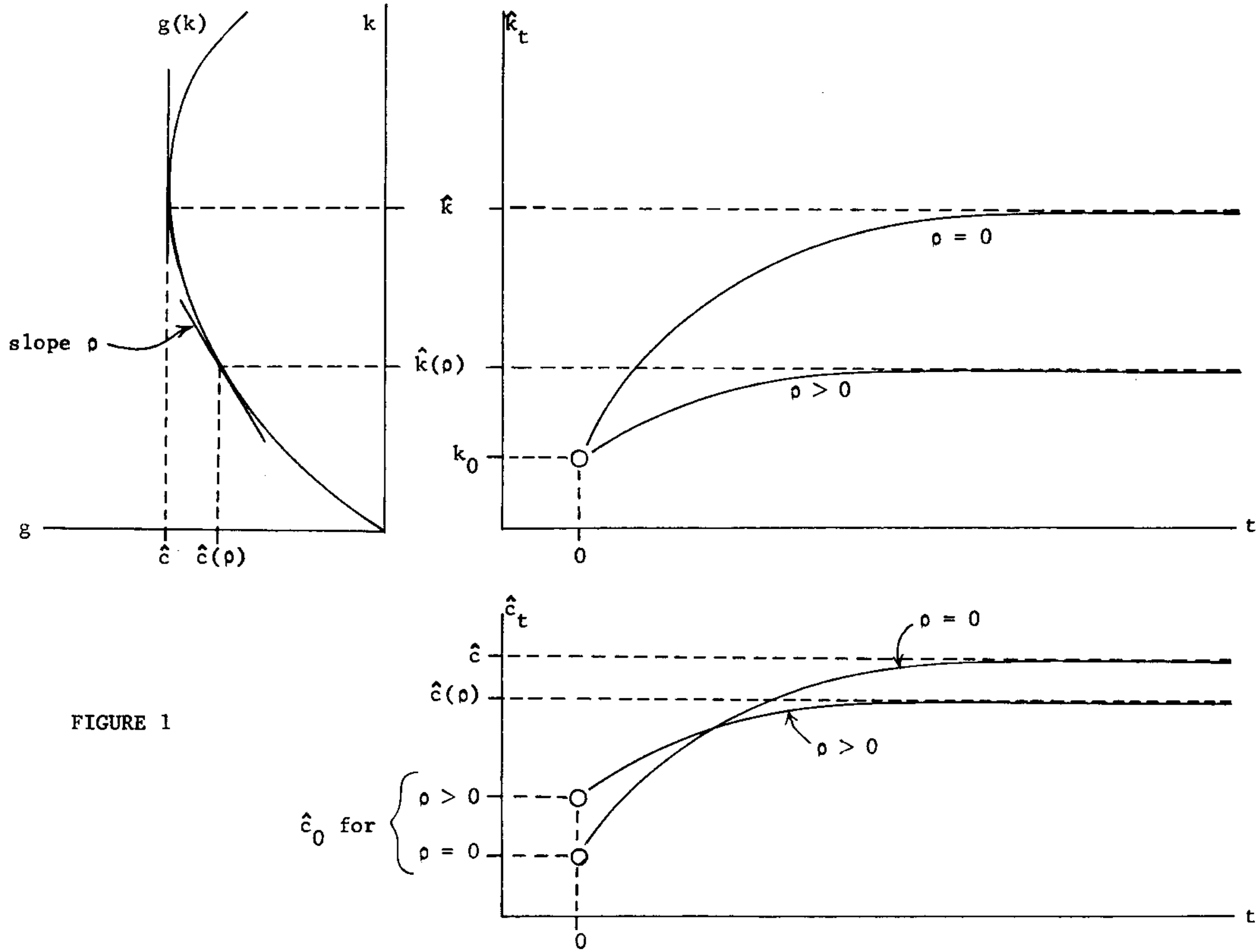


FIGURE 1

assumption of a constant population. An alternative but ultimately unsatisfactory assumption of exogenous exponential population growth at a rate λ allows $g(k)$ to be interpreted as the per capita output from a per capita capital stock k minus an allowance λk for the per capita investment needed just to make the total capital stock grow in proportion to the population.

3. The Exhaustible Resource Model

Hotelling's [1931] classic is for the theory of the optimal rate of utilization of exhaustible resources what Ramsey's paper is for the theory of optimal capital growth. Hotelling's context is that of a given demand function for a mineral resource. Separate sections deal with the allocation of the resource over time resulting from free competition, from a monopoly of supply, and from a consideration of the social optimum.

The optimal growth literature that has sprung up in the last two decades consists preponderantly of wide ramifications and generalizations of the Ramsey model. The attention given to the problem of best allocation of exhaustible resources over time has been much sparser. While resources have been introduced in an optimizing context in papers by Malinvaud [1953], Radner [1966, 1967] and Hansen and Koopmans [1972], it has been the resource flows rather than the total stocks that have been assumed exogenously given. This misses the main characteristic of exhaustible resources: that extraction and use can be shifted between future periods, subject to a _finite_ (though usually uncertain) upper bound on cumulative extraction over an infinite future. This stands in striking contrast to the fact that shiftability over time of consumption and capital formation is in the focus of the optimal growth literature.

A small step toward the recognition of exhaustible resources in optimal growth theory occurs in an article by Gale [1967] otherwise entirely in the Ramsey framework. On page 4 there is a brief "example 2" of the optimizing cake eater. Each day he can eat a piece of a size he chooses from a nonperishable cake of finite size, until it is exhausted. If his current utility from cake consumption r is an increasing and strictly concave function $v(r)$ of r, he seeks to maximize (using discrete time)

$$V \equiv \sum_{t=1}^{\infty} v(r_t) \tag{3.1}$$

subject to

$$r_t \geqq 0 \ , \quad \sum_{t=1}^{\infty} r_t \leqq R \ , \quad \text{say, where} \quad R > 0 \ . \tag{3.2}$$

Gale points out that no optimal program exists, because,

(1) if, for any t', t'' with $t' \neq t''$, we have $r_{t'} \neq r_{t''}$ in a given program $(r_t) = \{r_t | t = 1, 2, \dots\}$, then the program $(\bar{r}_t)$ with

$$\begin{cases} \bar{r}_t = \frac{1}{2}(r_{t'} + r_{t''}) & \text{for} \quad t = t', t'' \\ \bar{r}_t = r_t & \text{otherwise} \end{cases}$$

is better than (r_t) because of the strict concavity of $v(r)$,

(2) the only feasible program with $r_t = r_1$ for all t has $r_1 = 0$, which is clearly nonoptimal.

Hence in this problem a zero discount rate paralyzes the "optimizing" decision maker. (This may, but need not be the case with discounting at a positive rate, depending on the behavior of $v(r)$ as r approaches zero.)

Let us apply this simple model to Fisher's stranded sailors, with two modifications. First, rather than trading from private hoards, the sailors have pooled their resources and, having heard of Ramsey, wish their pooled stock R of hardtack to be allocated over time by maximization of a utility integral. Secondly, they are aware of the existence of a bare subsistence level $\underline{r}$ of consumption, at which survival is just possible, but below which all life ceases instantly. By instantly tightening the belt to a consumption level of $\underline{r}$, the group can assure itself a painful survival for $\bar{T} = R/\underline{r}$ days, but not longer. The paralysis of Gale's cake eater has now been avoided, but a new problem needs to be faced. It is possible to attain a higher level of daily consumption for the duration of survival by accepting a shorter period T of survival. Population P_t can therefore follow any of the following paths

$$\begin{cases} P_t = P_0 \,, & 0 \leqq t \leqq T \,, \\ P_t = 0 \,, & T < t \,, \end{cases}$$

where T is a decision variable constrained by $0 < T \leqq \bar{T}$.

This formulation raises a new question in the interpretation of the objective functional (1.1) of much optimal growth theory. As long as population change is treated as exogenously given, the ranking of feasible paths, and therefore the choice of an optimal path, are not changed if one adds a constant to the utility function, say

$$v^*(r) = v(r) + \varphi \ .$$

As soon as the number of people is no longer exogenously given for all times, $v(r)$ is pressed into the additional role of an <u>absolute</u> valuation placed on one day of life (group life in the present case) at the consumption level r . As a consequence, the addition of a constant to $v(r)$ will in general change the ranking of paths, hence also the optimal path. To simplify matters, we shall somewhat arbitrarily assign to life at the bare subsistence level $\underline{r}$ an intrinsic value of zero,

$$v(\underline{r}) = 0 \ , \tag{3.3}$$

although arguments for another number, positive or even negative, could be advanced. We now regard $v(r)$ as defined only for $r \geqq \underline{r}$.

Reverting to continuous time but still rejecting the discounting of future utilities we must then maximize

$$\int_0^T v(r_t)dt \tag{3.4}$$

with respect to (T, r_t) , subject to

$$0 < T \leqq \overline{T} \ , \quad \int_0^T r_t dt = R \ , \quad r_t \geqq \underline{r} \ . \tag{3.5}$$

Maintaining strict concavity of $v(r)$, optimality again requires constancy of r_t during survival, so that

$$r_t = r = R/T \quad \text{for} \quad 0 \leqq t \leqq T \ , \quad r_t = 0 \quad \text{for} \quad T < t \ .$$

Adopting r rather than $T = R/r$ as the remaining decision variable, we must now maximize

$$\int_0^T v(r_t)dt = \frac{R}{r}\, v(r)$$

over the domain of definition of $v(r)$. Since $v(r) > v(\underline{r}) = 0$ for $r > \underline{r}$, optimality requires $r > \underline{r}$, and $0 = \frac{d}{dr} \log\left(\frac{v(r)}{r}\right) = \frac{v'(r)}{v(r)} - \frac{1}{r}$, so

$$v'(r) = \frac{v(r)}{r} . \tag{3.6}$$

Figure 2 shows the construction of a unique optimal $\hat{r}$ by drawing a tangent from the origin to the curve $v = v(r)$. This construction leads to a unique and finite $\hat{r}$ if $v(\cdot)$ is bounded, and also if $\lim_{r\to\infty} v'(r) = 0$. (Functions $v(\cdot)$ for which no such $\hat{r}$ exists imply that the value of the maximand is higher the shorter the survival period—a case of little interest.) The "optimal" survival time is $\hat{T} = R/\hat{r}$, which, of course, falls short of the maximum survival time $\bar{T} = R/\underline{r}$.

What happens if we discount future utilities at the rate ρ, maximizing the welfare functional

$$\int_0^T e^{-\rho t} v(r_t)dt \tag{3.7}$$

with respect to (T, r_t), under the constraints

$$r_t \geqq \underline{r} \quad \text{for} \quad 0 < t < T , \quad \int_0^T r_t dt = R . \tag{3.8}$$

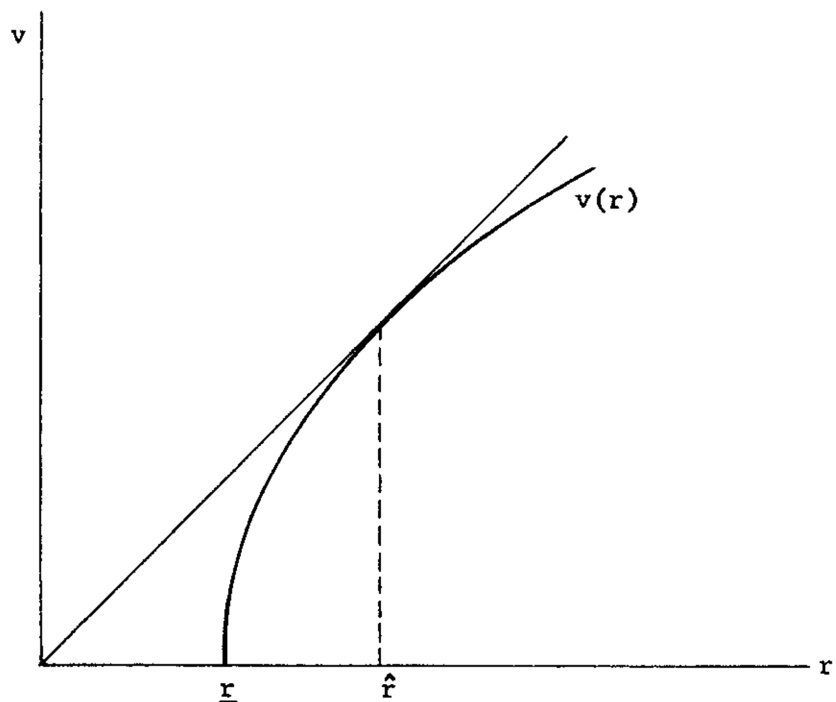


FIGURE 2

We give an intuitive argument, illustrated in Figure 3, suggesting that the optimal path is uniquely determined by three conditions.*

As a first step, hold T fixed at some arbitrarily chosen value with $0 < T < \bar{T}$, and maximize first only with respect to r_t. We must then have as the first, myopic, condition of "T-optimality" of r_t that

$$\varphi_t \equiv e^{-\rho t} v'(r_t) = \text{constant for } 0 \leqq t \leqq T . \tag{3.9}$$

This is so because, if $\varphi_{t'} > \varphi_{t''}$, say, for some t', t'' with $0 \leqq t', t'' \leqq T$, one could (see Figure 3) increase (3.7) by shifting a small amount of consumption from a neighborhood in $[0,T]$ of t'' to one of t'. (A proviso for this reasoning is that $r_t > \underline{r} + \epsilon > \underline{r}$ for all t with $0 \leqq t \leqq T$.)

Since $v''(r) < 0$, the condition (3.9) requires r_t to be a segment of a curve that is one of a family of descending curves. Any curve in this family can be identified by specifying the value of r_t for some suitable value of t. The second, terminal, condition** chooses the point $t = T$ for this purpose. Figure 3 shows another "small" modification of the path r_t that reduces the consumption flow on a short terminal interval $[T-\tau, T]$ by an amount ϵ and uses the amount $\tau\epsilon$ thus saved to extend the survival period by a flow at the constant level $r_t = r_T$ continuing for the period $[T, T+\tau']$, where $\tau' = \tau\epsilon/r_T$. The first order effect on the utility integral (3.7) is

---

*Reserving a rigorous proof for publication elsewhere.

**The form of this condition was suggested to me by William Nordhaus.

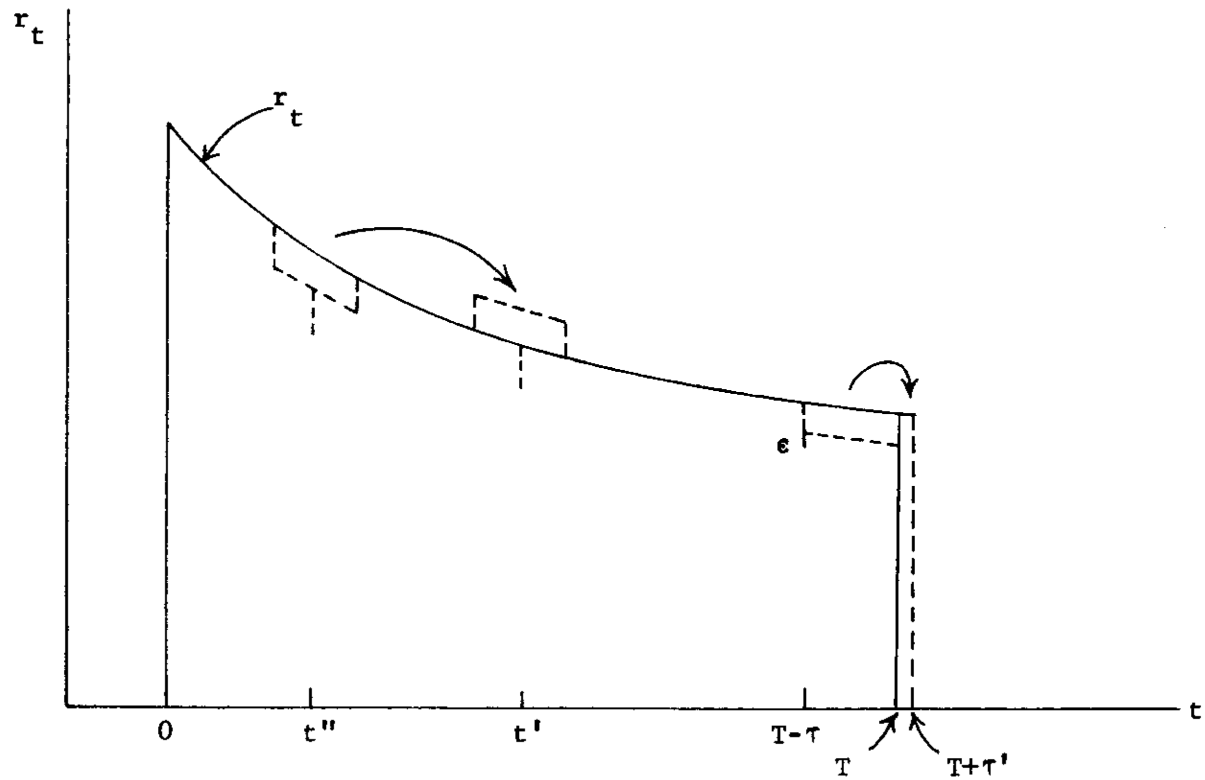


FIGURE 3

$$\left(-v'(r_T) + \frac{v(r_T)}{r_T}\right)\tau\epsilon e^{-\rho T} ,$$

where ϵ though small can be given either sign.* It follows that the segment r_t , $0 \leqq t \leqq T$, is anchored by the terminal condition

(3.10) $$r_T = \hat{r} ,$$

where $\hat{r}$ is the optimal consumption for $\rho = 0$ defined by (3.6). (Note that due to this construction the T-optimal path r_t satisfies the proviso stated above.)

Finally, the third, <u>length of survival</u>, condition uniquely determines the optimal survival time $\hat{T}(\rho)$ from the requirement that the segment $r_t = \hat{r}_t$ so selected shall just exhaust the given stock R .

Figure 4 indicates the nature of the optimal path $\hat{r}_t$, its connection with the utility function $v(\cdot)$, and its dependence on the discount rate ρ . The path $\hat{r}_t$ starts higher, descends faster, and ends sooner, the larger is ρ . <u>In the exhaustible resources model, discounting of future utilities favors an earlier generation over any surviving later generation, and shortens the period of survival</u>. <u>These effects are stronger the higher the discount rate</u>.

---

*For $\epsilon < 0$, one adds $|\epsilon|$ to r_t for $T' - \tau \leqq t \leqq T'$, where T' is just enough below T to again satisfy (3.8) with T' replacing T .

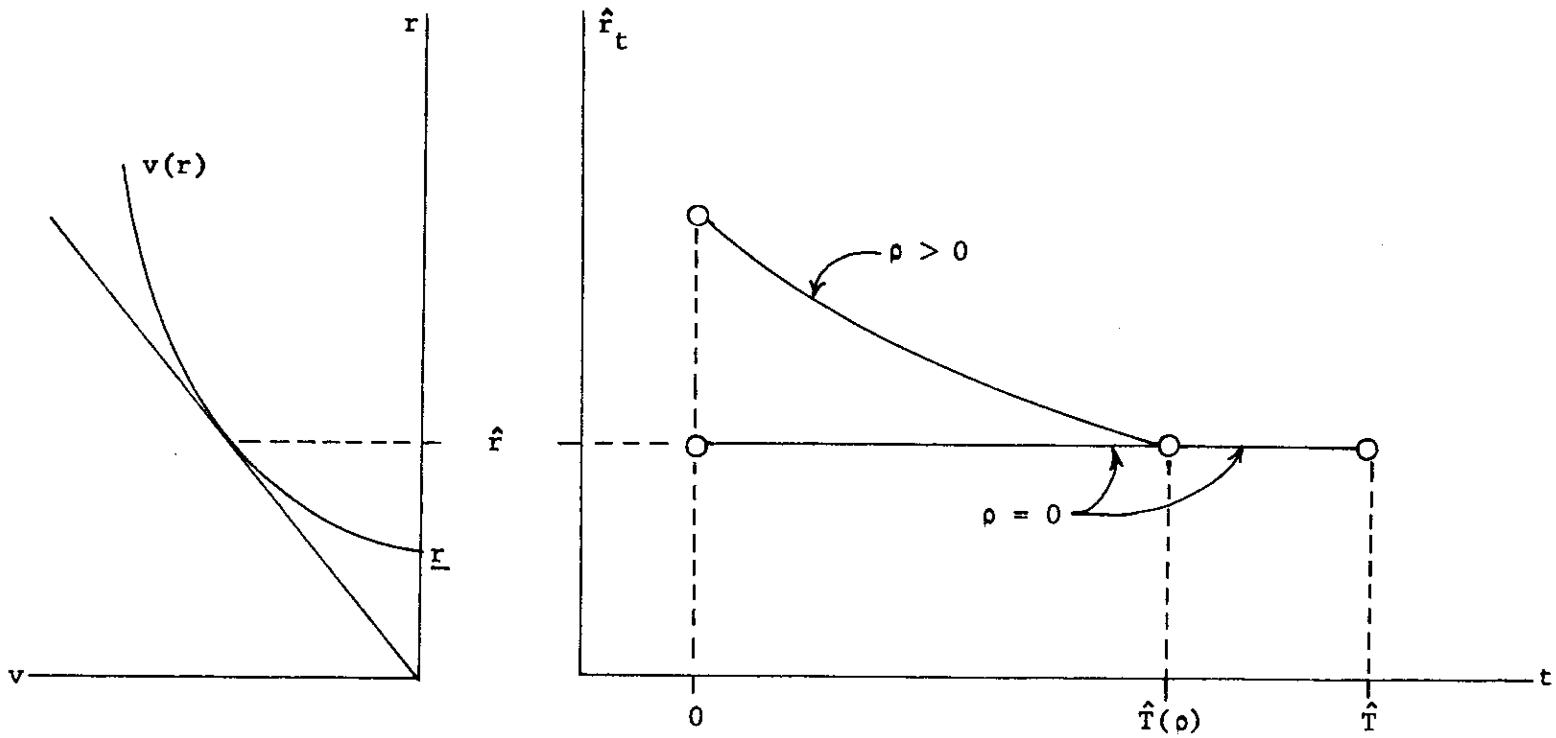


FIGURE 4

4. A Comparative Discussion

We shall now seek to interpret the contrasts in the solutions of the two models considered, and speculate about the empirical relevance of the various traits of the models.

Since the welfare functional is constructed along the same lines in the two cases, the contrast in the solutions must be due to the difference in supply conditions. In the capital model decision makers can at any time arrange to reap the prospective consumption-increasing benefits of a larger future capital stock only through net investment at points in time in a more nearby future. The future benefit to be derived from a unit of investment is smaller the larger the capital stock already attained. The optimal capital build-up therefore slows down as it proceeds, and stops at the point where the discounted cost of further investment balances the more strongly discounted benefit. The optimal rate of consumption therefore also increases with time, and slows down to a stop at a stationary consumption point corresponding to the stationary capital stock. The higher the discount rate, the lower both the stationary optimal capital stock and the associated consumption flow.

In contrast, in our simplified model of resource exhaustion, consumption of an extra unit of the resource can at any time be decided on and implemented instantly, without having to go through a process of prior investment. Moreover, the opportunity cost of this instant benefit is incurred later, by an equal curtailment of consumption within the survival period, or by a shortening of that period. Finally, the analysis leading to (3.6) shows that even in the absence of discounting a reduction in the rate of consumption below $\hat{r}$ is deemed too high a price for the extension of the survival period it could buy. Discounting can therefore only lift the earlier generations above the resource consumption levels of the later ones, along a path that stops when the level $\hat{r}$ is reached.

Let us now consider the question whether there is anything in the long-range outlook for natural resource availability that gives even a tinge of realism to the model of possibly slow but inexorable and ultimately catastrophic exhaustion.

This issue is the subject of a protracted and continuing debate between economists and conservationists. The view prevailing among economists is that technological change has kept enlarging the aggregate size and diversity of the economically accessible resource base faster than extraction diminishes the resource base. This enlargement has come about through cost reductions in extraction of a given material, and through substitution of more abundant materials for increasingly scarce ones, including the scientific discovery of entirely new, often plentiful, kinds of resources. These substitutions within the resources category may be accompanied by increased capital inputs needed for the utilization of the newer more abundant resources. In aggregate accounts comparing quantity indices of total output resource inputs and capital use inputs (all weighted by base-year prices), such substitutions may therefore show up as an aggregate substitution of capital for "resources," where the resource flow at the same time changes its composition. In contrast, within the resource sector, the chain of events whereby technological change dominates the effect of continuing specific resource extractions shows up in a gradual decrease in the labor and capital intput cost per unit of extractive output, relative to that same cost per unit of net output in the rest of the economy. A similar decrease is found in the market price of minerals relative to that of nonextractive goods, even though the price includes the element of scarcity rent not included in the cost measure.

This interpretation of events* is persuasively summarized in Chapter 1 of Barnett and Morse [1963] and statistically documented in Chapters 8 and 9. A more recent study by Nordhaus and Tobin [1972, pp. 14-17] extends their work by ingenious simulations [ibid., App. B, pp. 60-70] in which neoclassical aggregate production functions are employed to show that elasticities of substitution in excess of 1 between resources and "capital plus labor" best fit observed characteristics of aggregate economic growth in the period 1909-1958.

The forward-looking relevance of the empirical observations made by the Barnett-Morse and Nordhaus-Tobin teams depends on an extrapolation into the future of the observed resource saving or augmenting effects of technological change. Two observations are pertinent here.

The first concerns the protection of the environment, through suitable choices of new technology, and through the modification of existing technology especially when applied on a larger scale. Particular problems arise in the extraction of resources from deposits of increasing depths or lower grades. Regarded itself as a resource, the environment is exhaustible if subjected to irreversible damage, renewable if the damage is temporary or restorable at a cost. An aggregate evaluation of the cost

*An optimization model incorporating some of the traits of this interpretation was recently discussed by Kent Anderson [1972].

of protection of the environment must await further experience in the study and application of protective policies for many specific environmental problems. I shall here assume that a satisfactory level of protection can be attained over time at a reasonable cost.

The second observation concerns the need for looking behind the observed predominance of resource saving or augmenting technological change over resource extraction to seek to perceive the underlying causes. The first impression here is one of a huge reserve of detailed physical, chemical, geological and physiological relationships: known, suspected but not yet known, or even as yet unsuspected. The veil is gradually drawn away by a process of discovery, partly or initially accidental, gradually resulting from a more systematic search which is never assured of specific successes until these are actually achieved. Why should this process permit extrapolation of past aggregate relationships?

If present knowledge permits an answer to this question, the principal contributions should come from natural scientists and engineers. As long as there are many independent lines of advance for research and development, it is perhaps not unreasonable to assume that the proportion of successes will continue to fluctuate around the level of past experience, and a statistical aggregative approach would appear justified. The crucial question is whether perhaps there is at least one Achilles heel of resource supply, a specific resource that is in limited supply, essential to life and welfare, used dissipatively, and with no substitute in greater supply.

My rather casual inquiries and reading of scientific periodicals have not revealed a clear and present case of such an Achilles heel. The claim made by Goeller [1972] that the phosphorus-fertilizer-food chain might become an example if world population levels off only at a multiple of its present size has been contested by others on the basis (among other reasons) of the considerable abundance of phosphorus (see, for example, Wells [1973]). Another example might just possibly arise with regard to energy in the unlikely contingency that none of a substantial number of current or future largely independent R&D projects to widen the supply base of energy is really successful. _If_ the nuclear breeder should turn out not to be safe or otherwise not environmentally acceptable, _if_ controlled nuclear fusion should not be found workable on an industrial scale, _if_ both geothermal and solar energy use should turn out to remain limited to special and local situations, and _if_ no other new sources of energy are discovered and developed, ... only _then_ could our present stock of fossil and nuclear fuels become such an Achilles heel—though even then oil shale might extend the period of availability quite substantially.

An interesting case (though not an Achilles heel) brought out by Goeller is that of helium. Its cost of extraction is expected to go up by a large factor (Goeller mentions the number 100) when natural gas is exhausted and extraction shifts to the atmosphere as the source. While not regarded as essential to life, helium may well become more important than it is now if cryogenic power transmission is successfully developed, with important savings in transmission losses of energy.

If we were compelled to choose between the capital model and the exhaustible resource model, therefore, the former would seem as yet to have the greater relevance. In the next section we experiment with combinations of the two models.

5. Two Models Combining Capital and an Exhaustible Resource

We conclude by briefly considering two alternative models that combine the essential traits of the capital and resource models. The very simple form given to this combination will allow conclusions to be drawn almost directly from the results obtained for the two models in Sections 2 and 3 above.

In both models the welfare functional has the form

$$W = \int_0^T e^{-\rho t}(u(c_t) + v(r_t))dt \ . \tag{5.1}$$

The constraints common in form to both models are

$$c_t + \dot{k}_t = g(k_t) \ , \quad c_t, \ k_t \geqq 0 \ , \quad 0 < k_0 \quad \text{(given)} \tag{5.2}$$

$$\int_0^T r_t dt = R > 0 \ , \quad r_t \geqq \underline{r} \geqq 0 \ . \tag{5.3}$$

Thus the total utility flow is obtained additively from a flow $u(c_t)$ due to consumption c_t of a good produced with the use of a capital stock k_t and a constant labor force, along the lines of the Ramsey model, and a flow $v(r_t)$ arising from the rate of depletion r_t of a resource stock

R . (We neglect any capital and labor inputs involved in the extraction or consumption of the resource.) $u(\cdot)$ and $v(\cdot)$ are again increasing and strictly concave.

The two models differ only in regard to the specifications with regard to the horizon T and the resource flow utility function $v(\cdot)$. In the first model consumption of the resource is not essential to life. We can therefore take $T = \infty$, and specify that $\underline{r} = 0$, $v(0) = 0$, $v'(0) < \infty$. This model has some of the traits of the helium problem.

The problem now breaks up into two independent maximizations, of

$$(5.4) \qquad U = \int_0^\infty e^{-\rho t} u(c_t)dt \ , \qquad V = \int_0^\infty e^{-\rho t} v(r_t)dt \ ,$$

respectively. The maximization of U leads to the same optimal path $\hat{c}_t$ as before, while that of V has a somewhat different outcome only because now $\underline{r} = 0$, whereas survival can continue on the basis of the consumption flow c_t alone. The modification of Figure 2 thus needed is obvious. For the reasons given by Gale there is no optimal path for $\rho = 0$. For $\rho > 0$ there is an optimal path $r_t = \hat{r}_t$ defined by

$$(5.5) \qquad \begin{cases} e^{-\rho t} v'(r_t) = e^{-\rho T} v'(0) & \text{for} \quad 0 \leqq t \leqq T \ , \\ r_t = 0 \ , \quad \text{hence} \quad v(r_t) = 0 \ , & \text{for} \quad t \geqq T \ , \end{cases}$$

where we must again choose $T = \hat{T}(\rho)$ so that the resource constraint (5.3) is satisfied. Thus the resource flow diminishes over time, the more steeply the larger is ρ , and reaches zero at the time of exhaustion of the resource stock. The path $\hat{c}_t$ is not affected by these events, and is therefore independent of the resource stock R.

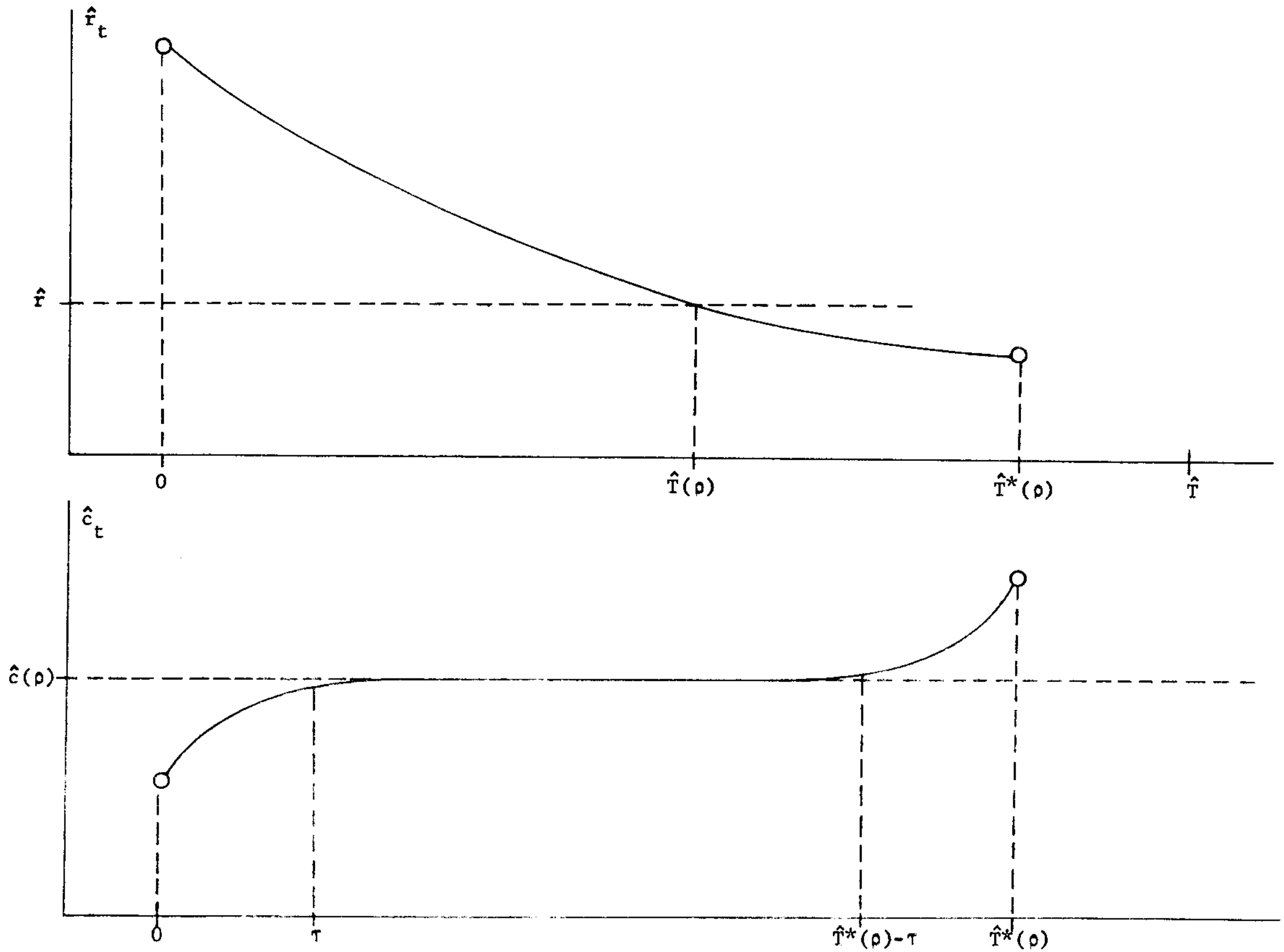
In the second model the resource is essential to life, and consumption of both c_t and r_t ceases at the time $t = T$ of exhaustion of the resource. Now, as in Section 3, $v(r)$ is defined only for $r \geqq \underline{r}$, where the minimum resource flow $\underline{r}$ required for survival satisfies $\underline{r} > 0$. Therefore, T in (5.1) is now a number such that $0 < T \leqq \overline{T} = R/\underline{r}$. We also specify $v(\underline{r}) = 0$, $\lim_{r \to \underline{r}} v'(r) = \infty$, as a further expression of the essential character of the resource. Finally, we set $u(0) = 0$, $\lim_{c \to 0} u'(c) = \infty$.

In this model we need to maximize $W = U_T + V_T$, where

$$(5.6) \qquad U_T = \int_0^T e^{-\rho t} u(c_t) dt \ , \qquad V_T = \int_0^T e^{-\rho t} v(r_t) dt \ ,$$

with respect to the triple (T, c_t, r_t), subject to the constraints (5.2), (5.3). The new element is that the same as yet unknown value of T has to occur in both U_T and V_T.

Figure 5 illustrates the nature of the unique optimal paths $\hat{c}_t$, $\hat{r}_t$ for $\rho > 0$. The resource use path $\hat{r}_t$ now dips below the value $\hat{r}$ near the end point $\hat{T}^*(\rho)$ of the survival period. This occurs because continued enjoyment of a high level $\hat{c}_t$ of general consumption compensates for what otherwise would have been an unacceptably low level of resource consumption $\hat{r}_t$. In an Appendix we give a list of intuitively plausible statements from which a proof of the assertions of this paragraph can be built up.



$\hat{r}_t$
$\hat{r}$
0
$\hat{T}(\rho)$
$\hat{T}^*(\rho)$
$\hat{T}$
t
$\hat{c}_t$
$\hat{c}(\rho)$
0
τ
$\hat{T}^*(\rho)-\tau$
$\hat{T}^*(\rho)$
t

FIGURE 5

An interesting implication of these findings is the behavior of the ratio q_t/p_t of the shadow prices

$$(5.7) \qquad q_t \equiv e^{-\rho t} v'(\hat{r}_t) = v'(\hat{r}_0) \ , \quad p_t \equiv e^{-\rho t} u'(\hat{c}_t) \ ,$$

of the resource and the consumption good on the long middle interval $[\tau, \hat{T}^*(\rho) - \tau]$ in the second model, and in the longer interval $[\tau, \infty)$ in the first. In both cases we have

$$(5.8) \qquad p_t \approx e^{-\rho t} u'(\hat{c}(\rho)) \ , \quad \text{hence} \quad q_t/p_t \approx \text{const.}\ e^{\rho t} \ .$$

Thus, the combination of discounting of future utilities at a positive rate and the costless storage in nature of the resource prior to extraction leads to an exponential increase in the shadow price of the resource relative to that of the consumption good. In a monetary system with the latter good as the numéraire, an ideal competitive market with indefinitely long foresight would therefore exhibit a sustained exponential increase in the scarcity price of the resource. The extent to which actual markets reflect this effect has been discussed by Nordhaus [1973].

APPENDIX

Statements Supporting the Solution of the Second Model of Section 5

As in Section 3, we consider again first a prescribed survival period T, which allows provisional independent maximizations of U_T and V_T for that period. The following intuitively plausible statements will not be sharpened and proved here. Those relating to U_T follow from the work of Brock [1971] and earlier work of Cass [1966] or Koopmans [1965]. Those relating to V_T are implied in the results of Section 3. Regarding U_T, writing $c_t = c_t^T$ for the T-optimal consumption path and $\hat{U}_T$ for the maximal value of U_T attained on that path, we have for sufficiently large T that

(i) c_t^T increases with t for $0 \leqq t \leqq T$,

(ii) c_0^T is practically independent of T and $\lim_{T \to \infty} c_0^T \equiv c_0^\infty > 0$,

(iii) c_t^T is close to $\hat{c}(\rho)$ during a long middle segment $[\tau, T-\tau]$, $0 < \tau \ll T$, of the survival period,

(iv) for t approaching T, c_t^T rises further while the associated capital stock k_t^T falls to $k_T^T = 0$,

(v) $\dfrac{d\hat{U}_T}{dT} > 0$ (for <u>all</u> $T > 0$),

(vi) $0 \leqq \lim_{T \to \infty} e^{\rho T} \dfrac{d\hat{U}_T}{dT} < \infty$ for all $\rho > 0$.

Regarding V_T , writing $r_t = r_t^T$ for the T-optimal path, $\hat{V}_T$ for the maximum attained, we have for $0 < T < \bar{T}$ that

(vii) r_t^T satisfies (3.9), hence r_t^T decreases as t increases and $r_t^T > \underline{r}$ for all $\rho > 0$ and $0 \leqq t \leqq T$,

(viii) $e^{\rho T} \cdot \frac{d\hat{V}_T}{dT} = v(r_T^T) - r_T^T v'(r_T^T) \left\{ \begin{matrix} > \\ = \\ < \end{matrix} \right\} 0$ for $T \left\{ \begin{matrix} < \\ = \\ > \end{matrix} \right\} \hat{T}(\rho)$ as defined following (3.10) and $e^{\rho T} \frac{d\hat{V}_T}{dT}$ decreases as T increases for $0 < T < \bar{T}$,

(ix) as $T \to \bar{T}$, $r_t^T \to \underline{r}$ uniformly in t for each ρ , hence

$$\lim_{T \to \bar{T}} e^{\rho T} \frac{d\hat{V}_T}{dT} = -\infty \ .$$

It follows from (v), (vi), (viii), (ix) that for any given $\rho > 0$, if the resource stock R and hence $\bar{T}$ are sufficiently large, W reaches a unique maximum for a survival period $T = \hat{T}^*(\rho)$, determined from the condition

$$e^{\rho T} \frac{d}{dT} (\hat{U}_T + \hat{V}_T) = 0 \ , \tag{A.1}$$

and located in the open interval

$$\hat{T}(\rho) < \hat{T}^*(\rho) < \bar{T} \ . \tag{A.2}$$

REFERENCES

Anderson, K., "Optimal Growth when the Stock of Resources in Finite and Depletable," Journal of Economic Theory, April 1972, pp. 256-267.

Barnett, H. J. and C. Morse, Scarcity and Growth, the Economics of Natural Resource Availability, Baltimore, Johns Hopkins Press, 1963.

Brock, W. A., "Sensitivity of Optimal Growth Paths with Respect to a Change in Target Stocks," in G. Bruckman and W. Weber, Eds., Contributions to the Von Neumann Growth Model, Zeitschrift für National-Ökonomice and Springer Verlag, New York and Vienna, 1971, pp. 73-89.

Cass, D., "Optimum Growth in an Aggregate Model of Capital Accumulation: A Turnpike Theorem," Econometrica, October 1966, pp. 833-850.

Fisher, Irving, The Theory of Interest, New York, Macmillan, 1930. Reprinted New York, Augustus Kelley, 1961.

Gale, D., "On Optimal Development in a Multi-Sector Economy," Review of Economic Studies, 34, January 1967, pp. 1-18.

Goeller, H. E., "The Ultimate Mineral Resource Situation—An Optimistic View," Proceedings, National Academy of Sciences, U.S.A., 69, October 1972, pp. 2991-92.

Hansen, T. and T. C. Koopmans, "On the Definition and Computation of a Capital Stock Invariant Under Optimization," Journal of Economic Theory, 5, December 1972, pp. 487-523.

Hotelling, H., "The Economics of Exhaustible Resources," Journal of Political Economy, 39, April 1939, pp. 137-75.

Koopmans, T. C., "On the Concept of Optimal Economic Growth," in The Econometric Approach to Development Planning, North Holland Publishing Co. and Rand McNally, 1966, and Pontificiae Academiae Scientiarum Scripta Varia, Vol. 28, 1965, pp. 225-300.

__________, "Intertemporal Distribution and 'Optimal' Aggregate Economic Growth," in Ten Economic Studies in the Tradition of Irving Fisher, New York, Wiley, 1967, pp. 95-126.

Malinvaud, E., "Capital Accumulation and Efficient Allocation of Resources," Econometrica, 21, April 1953, pp. 233-68.

Nordhaus, W. D., "Markets and Appripriable Resources," unpublished manuscript, 1973.

__________ and J. Tobin, "Is Growth Obsolete?" in Economic Growth, Fiftieth Anniversary Colloquium V, National Bureau of Economic Research, 1972, pp. 1-80.

Radner, R., "Optimal Growth in a Linear-Logarithmic Economy," International Economic Review, 7, January 1966, pp. 1-33.

__________, "Dynamic Programming of Economic Growth," Ch. 4 of Activity Analysis in the Theory of Growth and Planning, E. Malinvaud and M. O. L. Bacharach, Eds., New York, St. Martin's Press, 1967, pp. 111-41.

Ramsey, F. P., "A Mathematical Theory of Saving," Economic Journal, 38, December 1928, pp. 543-59.

Wells, F., "The Long Run Availability of Phosphorus," unpublished manuscript, February 14, 1973.